Linear Stability Analysis

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Consider the following ordinary differential equation (ODE):

\[
\frac{dx}{dt} = F(x) = ax + b \text{ for } t \geq 0,
\]

with the initial condition: \( x(t = 0) = x_0. \)

What can we learn out of this?
**The Exponential**

**general solution of Eq.(1)**

\[ x(t) = x_0 \exp\{at\} + \left( \frac{b}{a} \right) \left[ \exp\{at\} - 1 \right] \quad (2) \]

**stability: eigenvalue \( a \)**

\[ \lim_{t \to \infty} x(t) = \begin{cases} \infty ; & a \geq 0 \text{ unstable} \\ - \left( \frac{b}{a} \right) ; & a < 0 \text{ stable} \end{cases} \]

**steady state (for \( a < 0 \))**

\[ F(x^*) = 0 \implies x^* = -\frac{b}{a} \]
Objectives

- What happen when $F(x)$ is non-linear?

- How to deal with higher dimensional systems where $F(x)$ is now (either a linear or non-linear) matrix?
Consider the $n$–dimensional dynamical system, represented by the vector
\[
x = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix},
\]
and described by the ODE:
\[
\begin{align*}
\frac{dx_1}{dt} &= F_1(x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= F_2(x_1, x_2, \ldots, x_n) \\
&\quad \text{for } t \geq 0, \\
&\quad \text{(3)}
\end{align*}
\]
with the initial conditions, $x(t = 0) = x_0$, and $F(x)$ is a matrix nonlinear function of $x$.

**Aim:** use what we have learned from the exponential system
Taylor's Expansion

\[ F(x) \approx F(x^*) + \left. \frac{\partial F}{\partial x} \right|_{x=x^*} (x - x^*) + \cdots \] (4)

**fixed points**

\[
\begin{align*}
F(x^*) &= 0 \\ 
F_1(x_1, x_2, \ldots, x_n) &= 0 \\ 
F_2(x_1, x_2, \ldots, x_n) &= 0 \\ 
&\vdots \\ 
F_n(x_1, x_2, \ldots, x_n) &= 0
\end{align*}
\] (5)

**Jacobian matrix**

\[
J(x) = \begin{bmatrix}
\frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\
\frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\
& \vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n}
\end{bmatrix}
\] (6)
Linear approximation (close to fixed points)

Setting $\delta x = x - x^*$, the Eq.(3) becomes,

$$\frac{d(\delta x)}{dt} = J(x^*) \text{ for } t \geq 0,$$

(7)

with the initial conditions, $\delta x(t = 0) = x_0 - x^*$, and $J(x^*)$ is a constant matrix independent of $x$.

General solution (close to fixed points)

$$x(t) = x^* + \sum_{i=1}^{n} a_i v_i e^{\lambda_i t}; \quad \left\{ \begin{array}{l} \lambda_i = \text{eigenvalues} \\ v_i = \text{eigenvectors} \end{array} \right\} \text{ of } J(x^*)$$

(8)

$$\det |J(x^*) - \lambda 1| = 0; \text{ characteristic equation}$$

(9)

$a_i$ are constants determined by using the initial conditions.
Take home messages

About stability?

Procedure for stability analysis at fixed points

1. Determine the fixed point vector, $x^*$, solving $F(x^*) = 0$

2. Construct the Jacobian matrix, $J(x) = \frac{\partial F(x)}{\partial x}$

3. Compute eigenvalues of $J(x^*)$: $\det |J(x^*) - \lambda I| = 0$

4. Stability or instability of $x^*$ based on the real parts, $\Re(\lambda)$, of eigenvalues
What do eigenvalues tell us about stability?

**stability of** $x^*$ **based on the real parts of eigenvalues**

1. **all eigenvalues** have real parts less than zero $\implies x^*$ is stable
2. **at least one of the eigenvalues** has a real part greater than zero $\implies x^*$ is unstable
3. Otherwise: there is no conclusion (borderline case between stability and instability; require an investigation of the higher order terms)

**case of** $\lambda_m = 0$ $\implies$ **existence of an equilibrium**

$v_m = (v_{m,1}, v_{m,2}, \cdots, v_{m,n}) = \text{fraction of the system in each dimension}$
### What do eigenvalues tell us about stability?

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>All real and negative</td>
<td><img src="image1.png" alt="Graph" /></td>
</tr>
<tr>
<td>All real and one or more are positive</td>
<td><img src="image2.png" alt="Graph" /></td>
</tr>
<tr>
<td>All real eigenvalues are negative and there are imaginary parts</td>
<td><img src="image3.png" alt="Graph" /></td>
</tr>
<tr>
<td>One or more eigenvalues have a positive real part and there are imaginary parts</td>
<td><img src="image4.png" alt="Graph" /></td>
</tr>
<tr>
<td>Real parts of the eigenvalues are zero and there are imaginary parts</td>
<td><img src="image5.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

**Eigenvalues allow**

1. **stability analysis** of linear dynamical systems
2. **local** stability analysis of **nonlinear** dynamical systems
Procedure for $F(x) = ax + b$

1. Determine the fixed point, $x^*$, solving $F(x^*) = 0$

2. Construct the Jacobian, $J = \frac{dF(x)}{dx} = a$, constant

3. Compute **eigenvalues** of $J$: solving $J - \lambda = 0 \implies \lambda = J = a$

4. As $J = \text{constant}$, stability or instability of the whole system based on the sign of $\lambda$: $\lambda < 0 \implies$ the system is stable while $\lambda \geq 0 \implies$ the system is unstable
Example 1: Exponential decay

For the following kinetics described by the ODE below,

\[
\frac{dC}{dt} = f(C) = -k_{el} C + R \text{ for } t \geq 0,
\]

with the initial condition: \( C(t = 0) = C_0. \)

- **Fixed point = Steady State:** \( f(C^*) = 0 \implies C^* = \frac{R}{k_{el}} \)
- **Eigenvalue:** \( \lambda = -k_{el} < 0 \implies \text{stable system} \)

General solution of Eq.(10)

\[
C(t) = C^* + (C_0 - C^*) \exp\{-k_{el} t\}
\]
Procedure for $F(x) \neq ax + b$

1. Determine all fixed points, $x^*$, solving $F(x^*) = 0$

2. Construct the Jacobian, $J(x) = \frac{dF(x)}{dx}$

3. For each $x^*$, compute the eigenvalue: $\lambda = J(x^*)$

4. Conclude on the stability or instability of each $x^*$ based on the real parts of $\lambda$. 
Example 2: Modeling the population growth (P.-F. Verhulst, 1838)

Let $N$ represents the population size, the population growth is described by the Verhulst-Pearl equation:

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right)$$

(11)

where $r$ defines the growth rate and $K$ is the carrying capacity. Setting $x = N/K$ gives the differential equation (logistic equation),

$$\frac{dx}{dt} = rx(1 - x)$$

Pierre-Francois Verhulst

(1804 - 1849)
Example 2: The logistic growth

Consider the logistic differential equation:

\[ \frac{dx}{dt} = F(x) = rx(1 - x) \text{ for } t \geq 0 , \]

with the initial condition: \( x(t = 0) = x_0 \).

Aim: use the linear stability analysis
Example 2: The logistic growth: \( F(x) = rx(1 - x) \)

**fixed points of \( F(x) \)**

\[
F(x^*) = 0 \implies \begin{cases} 
  x_1^* = 0 \\
  x_2^* = 1
\end{cases} \quad (13)
\]

**Jacobian function**

\[
J(x) = \frac{dF}{dx} = r(1 - 2x) \quad (14)
\]

**eigenvalues of \( J(x) \)**

\[
\det |J(x^*) - \lambda I| = 0 \implies \lambda = \begin{cases} 
  J(x_1^*) = r ; & x_1^* = 0 \\
  J(x_2^*) = -r ; & x_2^* = 1
\end{cases} \quad (15)
\]
Example 2: The logistic growth

(l)inear approximation of the logistic

\[
\frac{d(x - x^*)}{dt} = \lambda(x - x^*) \quad ; \quad \lambda = \begin{cases} 
  r & ; \quad x \to x_1^* = 0 \quad \text{unstable} \\
  -r & ; \quad x \to x_2^* = 1 \quad \text{stable} 
\end{cases}
\]

- \( x \to x_1^* = 0 \), unstable: exponential growth
  \[ x(t) = x_0 e^{rt} \quad , \quad \text{(Malthus, 1798)} \quad (17) \]

- \( x \to x_2^* = x_{ss} = 1 \), stable: exponential decay
  \[ x(t) = x_{ss} + (x_0 - x_{ss}) e^{-rt} \quad (18) \]
Example 2: The logistic growth

**general solution of Eq. (12)**

\[
x(t) = \frac{x_0 e^{rt}}{1 + x_0 (e^{rt} - 1)} \quad (19)
\]

or,

\[
N(t) = \frac{K N_0 e^{rt}}{K + N_0 (e^{rt} - 1)} \quad (20)
\]

Example: \( N_0 = 1, \ r = 1.0 \) and \( K = 1,500 \).
Two Dimensions (two variables): Linear Systems

**Procedure for** \( F(x) = Jx \)

1. **Rewrite the system of ODE in the matrix form as**, \( \frac{dx}{dt} = Jx \)

2. **The Jacobian**, \( J = \frac{\partial F(x)}{\partial x} \), is constant and has the form:
   \[
   J = \begin{pmatrix}
   a & b \\
   c & d \\
   \end{pmatrix}
   \] (21)

3. **Compute **eigenvalues** of** \( J \) **as**:
   \[
   \text{det} | J - \lambda 1 | = 0 \implies \lambda^2 - (a + d)\lambda + ad - bc = 0 
   \] (22)
   \[
   \lambda_1, \lambda_2 = \frac{1}{2} \left[ a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)} \right] 
   \] (23)

4. **As** \( J = \text{constant} \): \( \Re(\text{of all } \lambda) < 0 \implies \) the system is stable while \( \Re(\text{of at least one } \lambda) \geq 0 \implies \) the system is unstable
Example 3: Pharmacokinetics

\[
\begin{align*}
\frac{dC_1}{dt} &= -(k_{12} + k_{el}) C_1 + k_{21} C_2 + R \\
\frac{dC_2}{dt} &= k_{12} C_1 - k_{21} C_2
\end{align*}
\]  

(24)

Q: develop the linear stability analysis
Two Dimensions (two variables): Linear Systems

Example 3: Pharmacokinetics

matrix $J$

$$J = \begin{bmatrix} -(k_{12} + k_{el}) & k_{21} \\ k_{12} & -k_{21} \end{bmatrix}$$  \hspace{1cm} (25)

eigenvalues of $J$

$$\begin{cases} \lambda_1 \\ \lambda_2 \end{cases} = -\frac{1}{2} \left[ k_{12} + k_{el} + k_{21} \pm \sqrt{(k_{12} + k_{el} + k_{21})^2 - 4(k_{12} + k_{el})k_{21} + 4k_{21}k_{12}} \right]$$ \hspace{1cm} (28)

Stability

$$\begin{cases} \lambda_1 \\ \lambda_2 \end{cases} = \Re(\lambda) \pm i\Im(\lambda)$$ \hspace{1cm} (26)

$$\Re(\lambda) \sim -(k_{12} + k_{el} + k_{21})$$ \hspace{1cm} (27)

$\implies$ stable system
Procedure for $F(x) \neq Jx$

1. Determine all fixed point vectors, $x^*$, solving $F(x^*) = 0$.

2. Construct the Jacobian matrix, $J(x) = \frac{\partial F(x)}{\partial x}$.

3. For each $x^*$, the Jacobian $J(x^*)$ has the form:

$$J(x^*) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

4. For each $x^*$, compute eigenvalues of $J(x^*)$ as:

$$\det |J(x^*) - \lambda 1| = 0 \implies \lambda^2 - (a + d)\lambda + ad - bc = 0$$

$$\lambda_1, \lambda_2 \right\} = \frac{1}{2} \left[ a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)} \right]$$

5. $\Re(\text{of all } \lambda) < 0 \implies x^*$ is stable while
$\Re(\text{of at least one } \lambda) \geq 0 \implies x^*$ is unstable.
Example 4: Lotka - Volterra Equation for Predator - Prey Systems

Let $x$ and $y$ represent the number of preys (e.g., rabbits) and of predators (e.g., foxes), respectively. The population dynamics is described by the system of Lotka-Volterra equations:

$$
\begin{align*}
\frac{dx}{dt} &= ax - bxy \\
\frac{dy}{dt} &= -cy + dxy
\end{align*}
$$

(32)

$a$ ($c$) is the net growth (death) rate of preys (predators) in the absence of predators (preys), $b$ the rate of predation affecting the prey population, and $d$ the growth rate of predators proportional to the food intake.

Q: develop the linear stability analysis
Two Dimensions (two variables): Non-Linear Systems

Example 4: Lotka - Volterra Equation for Predator - Prey Systems

fixed points of $F(u^*)$, $u^* = (x^*, y^*)$

\[
\begin{align*}
ax - bxy &= 0 \\
-cy + dxy &= 0
\end{align*}
\implies u_1^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } u_2^* = \begin{pmatrix} c/a \\ -d/b \end{pmatrix}
\] (33)

Jacobian $J(x)$

\[
J(x) = \begin{bmatrix}
a - by & -bx \\
dy & -c + dx
\end{bmatrix}
\] (34)
Example 4: Lotka - Volterra Equation for Predator - Prey Systems

Jacobian $J(x)$

$$J(x) = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix} \tag{35}$$

Stability analysis close to $u_1^*$

$$J(u_1^*) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \implies (a - \lambda)(c + \lambda) = 0 \implies \begin{cases} \lambda_1 = a \\ \lambda_2 = -c \end{cases} \tag{36}$$

$\lambda_1 > 0$ and $\lambda_2 < 0 \implies u_1^*$ is unstable
Two Dimensions (two variables): Non-Linear Systems

Example 4: Lotka - Volterra Equation for Predator - Prey Systems

Jacobian $J(x)$

$$J(x) = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}$$  \hspace{1cm} (37)

Stability analysis close to $u_2^*$

$$J(u_2^*) = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix} \implies \lambda^2 + ac = 0 \implies \begin{cases} \lambda_1 = -i\sqrt{ac} \\ \lambda_2 = i\sqrt{ac} \end{cases}$$ \hspace{1cm} (38)

$\lambda_1$ and $\lambda_2$ are imaginary $\implies u_1^*$ is oscillatory
Example 5: SIRS Epidemic model

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI + \gamma R \\
\frac{dI}{dt} &= \beta SI - \alpha I \\
\frac{dR}{dt} &= \alpha I - \gamma R
\end{align*}
\]

with \( S + I + R = N \) (39)

Q: develop the linear stability analysis and determine \( R_0 \)

Kermack & McKendrick
Example 5: SIRS Epidemic model

fixed points of $F(x^*)$, $x^* = (S^*, I^*, R^*)$

\[
\begin{align*}
-\beta SI + \gamma R &= 0 \\
\beta SI - \alpha I &= 0 \\
\alpha I - \gamma R &= 0
\end{align*}
\implies \quad x_1^* = \begin{pmatrix} N \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad x_2^* \neq x_1^*
\]

Determination of $R_0$

\[
x_2^* = \begin{bmatrix}
S_2^* = \frac{N}{R_0} \\
I_2^* = \left(\frac{\gamma}{\alpha + \gamma}\right) \left(\frac{R_0 - 1}{R_0}\right) N \\
R_2^* = \left(\frac{\alpha}{\alpha + \gamma}\right) \left(\frac{R_0 - 1}{R_0}\right) N
\end{bmatrix}
\]

where $R_0 = \frac{\beta N}{\alpha}$; $x_2^* > 0 \implies R_0 > 1$
Example 5: SIRS Epidemic model

Jacobian $J(x)$

$$J(x) = \begin{bmatrix} -\beta I & -\beta S & \gamma \\ \beta I & \beta S - \alpha & 0 \\ 0 & \alpha & -\gamma \end{bmatrix}$$  \hspace{1cm} (42)

Stability analysis close to $x_1^*$

$$J(x_1^*) = \begin{bmatrix} 0 & -\beta N & \gamma \\ 0 & \beta N - \alpha & 0 \\ 0 & \alpha & -\gamma \end{bmatrix} \implies \lambda(\gamma + \lambda)(\beta N - \alpha - \lambda) = 0 \implies \begin{cases} \lambda_1 = 0 \\ \lambda_2 = -\gamma \\ \lambda_3 = \alpha(R_0 - 1) \end{cases}$$  \hspace{1cm} (43)

$x_1^*$ is stable for $R_0 < 1$ and unstable for $R_0 > 1$
Three Dimensions (three variables): Non-Linear Systems

Example 5: SIRS Epidemic model

Jacobian $J(x)$

$$J(x) = \begin{bmatrix} -\beta I & -\beta S & \gamma \\ \beta I & \beta S - \alpha & 0 \\ 0 & \alpha & -\gamma \end{bmatrix}$$

(44)

Stability analysis close to $x_2^*$ (in the limit $R_0 > 1$)

$$J(x_2^*) = \begin{bmatrix} -\beta I_2^* & -\alpha & \gamma \\ \beta I_2^* & 0 & 0 \\ 0 & \alpha & -\gamma \end{bmatrix}$$

$$\lambda \left[ \lambda^2 + (\gamma + \beta I_2^*)\lambda + \alpha \beta I_2^* \right] = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = -\frac{1}{2} \left[ \gamma + \beta I_2^* + \sqrt{(\gamma + \beta I_2^*)^2 - 4\alpha \beta I_2^*} \right]$$

$$\lambda_3 = -\frac{1}{2} \left[ \gamma + \beta I_2^* - \sqrt{(\gamma + \beta I_2^*)^2 - 4\alpha \beta I_2^*} \right]$$

(45)

$\lambda_2$ and $\lambda_3$ are $< 0$ and $\lambda_1 = 0 \implies x_2^*$ is stable and steady state